Denseness of J= $\{A + BT: A, B \in M_k(Z)\}$ in the Space of all $k \times k$ real matrices, where T is a fixed $k \times k$ matrix with irrational entries

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Abstract- ***Several countable minimal dense subsets of R are known to exist. An important example is the set $Z + Zq = \{a + bq: a, b \in Z\}$, where q is an irrational number.In this paper, we establish an analogous result in the space of matrices $M_k(R)$. We prove that the set***
J= $\{A + BT: A, B \in M_k(Z)\}$ is dense in $M_k(R)$ under the metric on $M_k(R)$ defined by

$$d(X, Y) = \|X - Y\|_p = \left(\sum_{1 \le i,j \le k} |X(i,j) - Y(i,j)|^p\right)^{\frac{1}{p}}$$

Where T be a fixed matrix in $M_k(R)$ whose entries are irrational numbers.
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1. Introduction

The study of dense subsets of R under the usual metric has been a foundational topic in real analysis and number theory. It is well known that many dense subsets of R exist under the usual metric. The set of all rational numbers Q is a well-known example of a countable minimal dense subset of R. similarly, the set of all irrationals is also dense in R, though uncountable. Beyond these classical examples, the set $Z + Zq = \{a + bq: a, b \in Z\}$, where q is irrational,plays a central role in the construction of countable dense subsets of R. This set behaves similarly to Q and forms a minimal dense subset of R. Although several proofs exist for this result, here we present a sequential convergence proof that provides a constructive understanding of the denseness property. The proof uses a recursively defined sequence $(x_n) \subset Z + Zq$ converging to zero. Using this, we establish the denseness of $Z + Zq$ in R. We then extend this idea to the Euclidean space $\boldsymbol{R^m}$, and further to the matrix space $M_k(R)$, the set of all $k \times k$ real matrices. We prove that J= $\{A + BT: A, B \in M_k(Z)\}$ is dense in $M_k(R)$ whenever T is a matrix with irrational entries. This result provides a natural higher-dimensional generalization of the classical integer–irrational denseness property.

2. Preliminaries

Archimedean Property 2.1. [3] If $x, y \in R$ with $x > 0$, then there exist a positive integer n such that $nx > y$

Inequality Lemma 2.2. If $a, b > 0$ and $a + b = k$ $with$, $then$ $minimum\{a, b\} \le \frac{k}{2}$

Density Extension Lemma 2.3. Let A be a dense subset of R, then any set B with A$\subset B \subset R$ is also dense in R.

3. Denseness of $Z + Zq$ via Sequential Convergence and its extension

Theorem.3.1: Let Z is the set of all integers, and q is a fixed irrational. Then the set $Z + Zq = \{a + bq: a, b \in Z\}$is dense in R.

Proof: Let q be a fixed irrational number. We construct a strictly decreasing sequence (x_n) with $x_n \in Z + Zq$, $\forall n$ with $x_n \to 0$

Let $x_1 = q - [q]$, where $[q]$ denotes the greatest integer less than or equal to q

Clearly $0 < x_1 < 1$.

Let k_1 be the least positive integer such that $k_1\, x_1 > 1$.Then

$$0 < (k_1 - 1)\ x_1 < 1 < k_1\, x_1$$

By the Inequality Lemma, either $k_1\, x_1 - 1$ or $1 - (k_1 - 1)\ x_1$ less than or equal to $\frac{x_1}{2}$.(Atually it is strictly less than ,since q is irrational).Denote this smaller value by x_2

Then, $x_2 = min\{k_1\, x_1 - 1\, , 1 - (k_1 - 1)\ x_1\}$ and $0< x_2 \leq \frac{x_1}{2} < x_1 < 1$.Also $x_2 \in Z + Zq$

Proceeding recursively, we define $x_{n+1} = min\{k_n\, x_n - 1\, , 1 - (k_n - 1)\ x_n\}$ with k_n the least positive integer such that $k_n\, x_n > 1$.

Then ,$0< x_{n+1} \leq \frac{x_n}{2}$,so

$$0 < x_n \leq \frac{x_1}{2^{n-1}}$$

Implying $x_n \to 0$.

Hence, for any $\varepsilon > 0,$ *there exist* a positive integer N_0 such that

$$0< x_n < \varepsilon\, , for\ all\ n \geq N_0 \qquad .... \quad (1)$$

Next we will prove that the set $Z + Zq = \{a + bq:\ a, b \in Z\}$ is dense in R

Now let r be any real number, $\varepsilon > 0$ be given.

If $r \in Z + Zq$, there is nothing to prove.

So we assume $r \notin Z + Zq$.consider the case $r > 0$.

By the convergence of (x_n), we can choose an x_k in the sequence such that $0< x_k < r$ and $x_k < \varepsilon$.

That is, choose $x_k < min\{r, \varepsilon.\}$

Then there exist a positive integer M such that $Mx_k < r < (M + 1)x_k$.

Hence,

$r < (M + 1)x_k .= Mx_k + x_k < r + \varepsilon$, and since $(M + 1)x_k \in Z + Zq$

We have a point $(M + 1)x_k \in Z + Zq \cap B(r, \varepsilon)$, where $B(r, \varepsilon)$ denotes the ε −neibourhood of r

For $r < 0$, take $r = -s$ with $s > 0$,and $\varepsilon > 0$ be given.

Then by the above result, we can find a $y \in Z + Zq$ such that

$$s < y < s + \varepsilon$$

Then $$-s - \varepsilon < -y < -s$$

That is, $r - \varepsilon < -y < r$ and $-y \in Z + Zq$

$\Longrightarrow\ -y \in B(r, \varepsilon)$, with $-y \in Z + Zq$

This shows that for any real r and an $\varepsilon > 0$,the ball $B(r, \varepsilon)$ contains a point of $Z + Zq$

Thus $Z + Zq$ dense in R.

Theorem.3.2. Let $\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \dots, \mathrm{q}_\mathrm{k}$. be distinct irrationals. Then the set

Then the set $A = \{a + \sum_{i=1}^{k} a_i q_i :\ a, a_1, a_2, \dots, a_k \in Z\}$ is dense in R

Proof: $Z + Z\mathrm{q}_1 = \{a + b\mathrm{q}_1 :\ a, b \in Z\}$ is dense in R and $Z + Z\mathrm{q}_1 \subset A \subset R$, the result follows from the Density Extension Lemma

Theorem.3.3. Let $q = (\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \dots, \mathrm{q}_\mathrm{m}) \in\ R^m$ with $\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \dots, \mathrm{q}_\mathrm{m}$ are fixed irrationals. Then

$Z^m + Z^m q = \{(\mathrm{a}_1{+}\mathrm{b}_1\mathrm{q}_1, \mathrm{a}_2{+}\mathrm{b}_2\mathrm{q}_2, \dots, \mathrm{a}_\mathrm{m}{+}\mathrm{b}_\mathrm{m}\mathrm{q}_\mathrm{m}) : \mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{m}, \mathrm{b}_1, \mathrm{b}_2, \dots, \mathrm{b}_\mathrm{m} \in Z\}$ is dense in R^m under the metric

$$\|x - y\|_p = \ (\textstyle\sum_{i=1}^{m} |x_i - y_i|^p)^{\frac{1}{p}}$$

Proof: Let $x = (x_1, x_2, \dots, x_m) \in R^m$.

For each co-ordinate $x_k \in R$, by the above theorem there exist sequence $(\mathrm{a}_\mathrm{n}(\mathrm{k}) + \mathrm{b}_\mathrm{n}(\mathrm{k})\mathrm{q}_\mathrm{k})$ in $Z + Zq_k$ such that

$$\mathrm{a}_\mathrm{n}(\mathrm{k}) + \mathrm{b}_\mathrm{n}(\mathrm{k})\mathrm{q}_\mathrm{k} \to x_k$$

Hence,

$$(\mathrm{a}_\mathrm{n}(1) + \mathrm{b}_\mathrm{n}(1)\mathrm{q}_1, \mathrm{a}_\mathrm{n}(2) + \mathrm{b}_\mathrm{n}(2)\mathrm{q}_2, \dots, \mathrm{a}_\mathrm{n}(\mathrm{m}) + \mathrm{b}_\mathrm{n}(\mathrm{m})\mathrm{q}_\mathrm{m}) \to (x_1, x_2, \dots, x_m)$$

That is $a_n + b_n q \to x$, where $a_n = \big(a_n(1), a_n(2), \dots, a_n(m)\big)$, $b_n = \big(b_n(1), b_n(2), \dots, b_n(m)\big) \in R^m$.This shows that $Z^m + Z^m q$ is dense in R^m under the metric

$$d(x, y) = \|x - y\|_p = \left(\sum_{i=1}^{m} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

4. Extension to Space of Matrices

Theorem.4.1. Let T$\in\ M_k(I)\ be\ fixed$,where I is the set of all irrationals. Then for $any\ X \in M_k(R)$, there exist sequences $(A_n), (B_n)$ with $A_n, B_n \in\ M_k(Z) such\ that\ \ A_n + B_n T \to X$ under the metric on $M_k(R)$ defined by

$$d(X, Y) = \|X - Y\|_p = \left(\sum_{1 \le i, j \le k} |X(i, j) - Y(i, j)|^p\right)^{\frac{1}{p}}$$

Hence the set J= $\{A + BT : A, B \in M_k(Z)\}$ is dense in $M_k(R)$.This set is also a countable dense subset of $M_k(R)$

Proof: Let $X = [X(i, j)] \in\ M_k(R)$ and T= $[T(i, j)]$

We have the set $A = \{a + \sum_{m=1}^{k} a_m q_m :\ \ a, a_1, a_2, \dots, a_k \in Z\}$, where $\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \dots, \mathrm{q}_\mathrm{k}$ are fixed irrationals, is dense in R

That is,for each $entry\ X(i, j)$ of X , there exist sequences $(A_n(i, j))$, $(B_n(i, j))$ with entries in Z such that

$C_n(i,j) = A_n(i,j) + \sum_{m=1}^{k} B_n(i,m)T(m,j) \to X(i,j)$
That is, for each pair (i,j) *and* a given $\varepsilon > 0$, there exist a positive integer N such that

$$\left|A_n(i,j) + \sum_{m=1}^{k} B_n(i,m)T(m,j) - X(i,j)\right| < \frac{\varepsilon}{k^2} \ ,for\ every\ n \geq N \text{....(1)}$$

Take the matrices $A_n = [A_n(i,j)\,]$, $B_n = [B_n(i,j)]$. Then, $A_n + B_nT = [C_n(i,j)]$ such that

$$\|\,(A_n + B_nT) - X\|_p = \left(\sum_{1\leq i,j\leq k} |C_n(i,j) - X(i,j)|^p\right)^{\frac{1}{p}}$$

$$< \left(\sum_{1\leq i,j\leq k} \left|\frac{\varepsilon}{k^2}\right|^p\right)^{\frac{1}{p}} = \frac{\varepsilon}{k^2}.k^2 = \varepsilon \ ,for\ every\ n \geq N.$$

Thus the sequence$(A_n + B_nT)$ converges to X, proving that J is dense in $Mk(R)$. ■

5. Conclusions: We established the denseness of the set $Z + Zq$ via a sequential convergence argument. The result was extended to $R^m$and further to the space of matrices $Mk(R)$.The construction shows that the set J= $\{A + BT : A, B \in M_k(Z)\}$ is a countable minimal dense subset of $M_k(R)$ under the p-norm metric.

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