A Special Class of Multi-Fuzzy Algebra Over Multi-Fuzzy Field

K.Shaju[1], Sabu Sebastian[2], P.Priyanka[3], C.Haseena[4], I.Gafoor[5] and S.J.Sangeeth[6]
[1]Research Scholar,Department of Mathematical Sciences,Kannur University,Kerala,India
E-mail: shaju.kollaroth@gmail.com
[2]Professor, Department of Mathematics, Nirmalagiri College, Kerala,India
E-mail:sabukannur@gmail.com
[3,4,5,6]Research Scholar,Department of Mathematical Sciences,Kannur University,Kerala,India

Abstract- The concept of multi-fuzzy algebra extends the traditional notions of fuzzy algebra by introducing multi-dimensional membership functions, allowing for a more nuanced representation of uncertainty and vagueness in algebraic structures. This paper explores the properties and algebraic structures of multi-fuzzy algebras over multi-fuzzy field, generalizing the framework of fuzzy vector spaces and fuzzy fields. We introduce a novel approach using a bridge function h to define multi-fuzzy algebras when the dimensions of the multi-fuzzy set on an algebra A and the multi-fuzzy field differ. The paper investigates the influence of this bridge function on the algebraic properties of these spaces and provides a comprehensive analysis of their structure. Key results include the characterization of multi-fuzzy algebras, their intersections, and their behaviour under linear transformations

Keywords: Multi-fuzzy vector spaces, Multi-fuzzy field, Multi-fuzzy Algebra, Space of Multi-fuzzy Algebra over Multi-fuzzy field with respect to a bridge function

1. Introduction

Fuzzy set theory, introduced by Zadeh in 1965, has revolutionized the way we model uncertainty and imprecision in mathematical structures. Over the years, this theory has been extended to various algebraic structures, including fuzzy groups, fuzzy rings, and fuzzy vector spaces. The concept of fuzzy vector spaces over fuzzy fields, introduced by Nanda, further generalized the idea of vector spaces by incorporating fuzzy membership functions. The notion of multi-fuzzy sets was introduced by Sabu Sebastian and T.V. Ramakrishnan in 2010, allowing for multi-dimensional membership functions that capture more complex forms of uncertainty.

This paper aims to extend the theory of fuzzy algebra by introducing multi-fuzzy algebras over multi-fuzzy field. We explore the properties of these algebras, particularly when the dimensions of the multi-fuzzy set on an algebra and the multi-fuzzy field on the field are equal. Additionally, we introduce a bridge function h to handle cases where the dimensions differ, providing a more flexible framework for defining multi-fuzzy algebras. The paper investigates the algebraic structure of these spaces, focusing on the role of the bridge function in determining their properties.

The primary contributions of this work include:

1. A formal definition of multi-fuzzy algebra over a multi-fuzzy field.
2. The introduction of a new type of multi-fuzzy algebra over a multi-fuzzy field with respect to a bridge function h when dimensions of multi-fuzzy set on the algebra and multi-fuzzy field are differ.
3. A detailed analysis of the algebraic properties of these spaces, including their intersections and behaviour under linear transformations.
4. Some theorems characterizing the structure of multi-fuzzy algebras and their relationship to sub algebras.

This research not only generalizes existing theories but also opens new avenues for exploring multi-fuzzy linear algebra and its applications in areas such as decision-making, pattern recognition, and artificial intelligence.

2. Preliminaries

Definition 2.1. Let $\{\langle L_i, \vee_i, \wedge_i \rangle, 1 \le i \le k\}$ be a family of complete distributive lattices. Then the product $L = \prod_{i=1}^{k} L_i$ forms a lattice with *supremum* $\vee$ *and infimum* $\wedge$, defined by:

$$(a_1, a_2, \dots, a_k) \vee (b_1, b_2, \dots, b_k) = (a_1 \vee_1 b_1, a_2 \vee_2 b_2, \dots, a_k \vee_k b_k)$$

$$(a_1, a_2, \dots, a_k) \wedge (b_1, b_2, \dots, b_k) = (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2, \dots, a_k \wedge_k b_k).$$

Remark: In all cases we consider each lattice to be a complete distributive lattice. The supremum and infimum elements of the lattice L are denoted by 1_L and 0_Lrespectively.

Definition 2.2[2].*Let* $L = \prod_{j=1}^{k} L_j$ be a product complete distributive lattices. Let $t_j: L_j \times L_j \to L_j$ *for* $j = 1,2, \dots, k$ be t-norms. The T- operator (or T-norm) T:$L \times L \to L$ is defined by:

$T(x, y) = (t_1(x_1, y_1), t_2(x_2, y_2), \dots, t_k(x_k, y_k))$

where $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k) \in L$

Let $s_j: L_j \times L_j \to L_j$ *for* $j = 1,2, \dots, k$ be t-co-norms.

The **S-** operator **S:**$L \times L \to L$ is defined by:

$S(x, y) = (s_1(x_1, y_1), s_2(x_2, y_2), \dots, s_k(x_k, y_k))$

We denote $T = (t_1, t_2, \dots, t_k)$ and $S = (s_1, s_2, \dots, s_k)$.

Definition 2.3[1]: Let X be a set, and let $L = \prod_{j=1}^{k} L_j$ be a product of lattices. Let k be a positive integer. A multi-fuzzy set A in X is a set of ordered (k+ 1)- tuples:

$A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)): x \in X\}$,where $\mu_j \in {L_j}^X, 1 \le j \le k$. The function $\mu = (\mu_1, \mu_2, \dots, \mu_k): X \to L$ is called multi-fuzzy membership function, and k is called the dimension of A (or the dimension of μ). The set of all multi-fuzzy sets of X to L is denoted by $M^k FS(X, L)$.

Definition 2.4. Let $L = \prod_{j=1}^{k} L_j$ be a product complete distributive lattices and $T = (t_1, t_2, \dots, t_k)$ is a T-norm on L. Let F(+,.) be a field. A multi fuzzy set $A = \{(p, \lambda_1(p), \lambda_2(p), \dots, \lambda_k(p)): p \in \mathbb{F}\}$, where $\lambda_j \epsilon {L_j}^F, 1 \le j \le k$, with membership function $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ *is* called a multi fuzzy field of F under T-norm T if for all $p, q \ \epsilon F$:

a) $\lambda(p + q) \ge T(\lambda(p), \lambda(q)) = (t_1(\lambda_1(p), \lambda_1(q), t_2(\lambda_2(p), \lambda_2(q)), \dots, t_k(\lambda_k(p), \lambda_k(q)))$.

b) $\lambda(-p) \ge \lambda(p)$, for all p *in* F.

c) $\lambda(pq) \ge T(\lambda(p), \lambda(q)) = (t_1(\lambda_1(p), \lambda_1(q)), t_2(\lambda_2(p), \lambda_2(q)), \dots, t_k(\lambda_k(p), \lambda_k(q)))$.

d) $\lambda(p^{-1}) \ge \lambda(p)$, for all non zero element $p \epsilon F$.

e)$\lambda(0_F) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k}) = \lambda(1_F)$,

where 0_F is the zero element and 1_F is the unity in F.$1_{L_j}$is the supremum of L_j for each $1 \le j \le k$ so that 1_L is supremum of L.

The set all multi-fuzzy field of F under T-norm $T = (t_1, t_2, \dots, t_k)$ is denoted by $MFF^k(F, L, T)$

We say that $\lambda = (\lambda_{1,}, \lambda_2, \ldots, \lambda_k)$ is a multi-fuzzy field of F and write $\lambda \epsilon MFF^k(F, L, T)$.

If T is the minimum t-norm, then $MFF^k(F, L, T)$ is denoted by $MFF^k(F, L)$

3.Multi-fuzzy algebra over multi-fuzzy field based under T-norm

Definition 3.1: Let $L = \prod_{j=1}^{k} L_j$ be a product of lattices, and let $T = (t_{1,}, t_2, \ldots, t_k)$ be a T-norm on L. Let A be an associative, distributive, commutative algebra over the field F, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \epsilon MFF^k(F, L, T)$.A multi-fuzzy set B=$\{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) : x \in A\}$, where $\mu_j \epsilon L_j{}^A$ for $1 \le j \le k$,with membership function $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, is called a multi-fuzzy algebra over the multi-fuzzy field λ of F under T-norm T, if it satisfies the following conditions for all $x, y \,\epsilon A$ and $p \epsilon F$:

a) $\mu\,(x + y) \ \ge \mathrm{T}(\,\mu(x), \mu(y))$

b) $\mu(-x) \ge\ \mu(x)$, for all $x\,\epsilon A$

c). $\mu(p.x) \ge \mathrm{T}(\,\lambda(p), \mu(x))$

d) $\mu(xy) \ \ge \mathrm{T}(\,\mu(x), \mu(y))$

e) $\mu(0_A) = 1_L = (1_{L_1}, 1_{L_2}, \ldots, 1_{L_k})$,where 0_A is the zero vector in A, and 1_L is the largest element in L.

The set all multi-fuzzy algebra of A over the multi-fuzzy field λ of F under the T-norm T is denoted by $MFA^k(A, \lambda, L, T)$.If the T-norm is the minimum, we denote it by $MFA^k(A, \lambda, L)$.

We simply say that $\mu = (\mu_{1,}, \mu_2, \ldots, \mu_k)$ is a multi-fuzzy algebra of A over the multi-fuzzy field λ of F,and denote it as $\mu \epsilon MFA^k(A, \lambda, L, T)$.

Remark 3.1: If we consider $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ as a multi- fuzzy algebra over the ordinary field F instead of the multi-fuzzy field $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of F, where $\lambda(p) = 1_L = (1_{L_1}, 1_{L_2}, \ldots, 1_{L_k})$, then condition (c) reduces to, $\mu(p.x) = \mu(x)$.

The set of all multi- fuzzy algebra of A over the ordinary field F under T-norm T is denoted by $MFA^k(A, L, T)$

Remark.3.2: In all cases, A is assumed to be an associative, distributive, and commutative algebra over F.

Theorem.3.1. Let A be an algebra over the field F and Let μ be a multi-fuzzy subset of A with $\mu(0_A) = 1_L$.Then:

1) $\mu \epsilon MFA^k(A, \lambda, L, T)$ if and only if, for all $x, y\ \epsilon\ A$ and all $p, q \epsilon F$:

   i) $\mu\,(px + qy) \ge \mathrm{T}(\mathrm{T}(\,\lambda(p), \mu(x)), \mathrm{T}(\,\lambda(q), \mu(y)))$

   ii) $\mu\,(xy) \ge \mathrm{T}(\,\mu(x), \mu(y))$

2) $\mu \epsilon MFA^k(A, L, T)$if and only if, for all $x, y\ \epsilon\ A$ and $all\ p, q \epsilon F$:

   i) $\mu\,(px + qy) \ge \mathrm{T}(\ \mu(x), \mu(y))$

   ii) $\mu\,(xy) \ge \mathrm{T}(\,\mu(x), \mu(y))$.

Theorem.3.2. Let A be an algebra over the field F,and let $\mu, \tau \epsilon MFA^k(A, \lambda, L, T)$.Then their intersection

$\mu \cap \tau \epsilon MFA^k(A,\lambda,L,T)$.

Definition 3.2(Multi-Fuzzy Algebra Over a Multi-Fuzzy Field Based on Lattice Operations)

Let $L = \prod_{j=1}^{k} L_j$ be a product of lattices. Let A be an associative,distributive and commutative algebra over the field F, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \epsilon MFF^k(F,L)$.A multi-fuzzy set B=$\{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) : x \in A\}$, where $\mu_j \epsilon {L_j}^A$ for $1 \le j \le k$,with membership function μ =$(\mu_1, \mu_2, \ldots, \mu_k)$, is called a multi-fuzzy algebra over the multi-fuzzy field λ of F based on lattice operations if,for all $x, y \epsilon A$ and $p \epsilon F$,we have:

1. $\mu(x+y) \ge \mu(x) \wedge \mu(y) = \left(\mu_1(x) \wedge_1 \mu_1(y),\ \mu_2(x) \wedge_2 \mu_2(y), \ldots, \mu_k(x) \wedge_k \mu_k(y)\right)$
2. $\mu(-x) \ge \mu(x)$, for all $x \epsilon A$
3. $\mu(p.x) \ge \lambda(p) \wedge \mu(x) = \left((\lambda_1(p) \wedge_1 \mu_1(x)), (\lambda_2(p) \wedge_2 \mu_2(x)), \ldots, (\lambda_k(p) \wedge_k \mu_k(x))\right)$
4. $\mu(xy) \ge \mu(x) \wedge \mu(y) = \left(\mu_1(x) \wedge_1 \mu_1(y),\ \mu_2(x) \wedge_2 \mu_2(y), \ldots, \mu_k(x) \wedge_k \mu_k(y)\right)$
5. $\mu(0_A) = 1_L = (1_{L_1}, 1_{L_2}, \ldots, 1_{L_k})$, where 0_A is the zero vector in A and 1_L is the largest element in L

This means μ is a multi-fuzzy-vector space over λ with an additional property (4).

The set of all multi-algebra of A over the multi-fuzzy field λ of F, based on lattice operations of $L = \prod_{j=1}^{k} L_j$ is denoted by $MFA^k(A,\lambda,L)$.

We say that $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, is a multi- algebra of A over the multi-fuzzy field λ of F and denote it as $\mu \epsilon MFA^k(A,\lambda,L)$.

Theorem 3.3. Let $\{\mu_j \epsilon MFA^k(A,\lambda,L) : j \in J\}$ be an indexed family of multi-fuzzy algebras of A over the multi-fuzzy field λ of F.Then their intersection $\mu = \bigcap_{j \in J} \mu_j \epsilon MFA^k(A,\lambda,L)$.

Theorem 3.4. Let A be an algebra over the field F and let $\mu, \tau \in MFA^k(A,\lambda,L))$.

Then, the multi-fuzzy set $\mu \times \tau : A \times A \to L$

$(\mu \times \tau)(z_1, z_2) = \inf\{\mu(z_1), \tau(z_2)\}$ for all $(z_1, z_2) \in A \times A$, is a multi-fuzzy algebra of $A \times A$ over λ,that is,

$$\mu \times \tau \in MFA^k(A \times A, \lambda, L).$$

4. Multi-Fuzzy Algebra μ over Multi-Fuzzy Field λ with dim $\mu \neq$dim λ

Definition 4.1. Let $M = \prod_{i=1}^{m} M_i$ and $L = \prod_{j=1}^{k} L_j$ be products of lattices. Let A be the multi-fuzzy set A=$\{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) : x \in X\}$, $where\ \mu_j \epsilon {L_j}^X for\ 1 \le j \le k$, with membership function μ =$(\mu_1, \mu_2, \ldots, \mu_k)$ defined on a set X with value domain $L = \prod_{j=1}^{k} L_j. Let\ h: L \to M$ be a crisp function.

The multi-fuzzy set B of X is defined by:

B=$\left\{\left(x, \mu^h(x)\right) : x \in X\right\}$, where the membership function $\mu^h : X \to \prod_{i=1}^{m} M_i$ is given by:

$$\mu^h(x) = h(\mu(x)), \text{for all } x \in X$$

The function μ^h is called fuzzy transformation of multi-fuzzy set μ with respect to the crisp function h.

Clearly, $\mu \epsilon M^k FS(X,L)$ and $\mu^h \in M^m FS(X,M)$.

Theorem.4.1. Let $\mu, \tau \in M^k FS(X,L)$ and $h: L \to M$ be a monotonic function. Then:

a) If h increasing and $\mu \subseteq \tau$,then $\mu^h \subseteq \tau^h$.
b) If h is decreasing and $\mu \subseteq \tau$,then $\tau^h \subseteq \mu^h$
c) If h is increasing ,then $(\mu \cap \tau)^h \subseteq \mu^h \cap \tau^h$
d) If h is increasing ,then $\mu^h \cup \tau^h \subseteq (\mu \cup \tau)^h$

Definition.4.2.(Multi fuzzy vector space μ over multi fuzzy field λ with dim $\mu \neq$ dim λ)

Let $M = \prod_{i=1}^m M_i$ and, $L = \prod_{j=1}^k L_j$ be products of complete distributive lattices. Let A be an algebra over the field F, and $\lambda = (\lambda_{1,}, \lambda_2, \dots, \lambda_m) \epsilon MFF^m(F, M, U)$. Define a multi-fuzzy set B by: B=$\{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)): x \in A\}$, *where* $\mu_j \epsilon {L_j}^A, 1 \leq j \leq k$, and the membership function is $\mu = (\mu_{1,}, \mu_2, \dots, \mu_k)$ .If $k \neq m$ and let $h: M \to L$ is a crisp function,then μ is called a multi-fuzzy algebra over the multi-fuzzy field λ of F under T-norm T (defined on L) with respect to the function h, if for all $x, y \epsilon A$ and $p \epsilon F$,the following conditions hold:

1. $\mu(x+y) \geq \mathrm{T}(\mu(x), \mu(y))$

2. $\mu(-x) \geq \mu(x)$, For all $x \epsilon A$

3. $\mu(p.x) \geq \mathrm{T}\left(\lambda^h(p)), \mu(x)\right)$

4. $\mu(xy) \geq \mathrm{T}(\mu(x), \mu(y))$

5. $\mu(0_A) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k})$,

where 0_A is the zero vector in A, and 1_L is the supremum element in L.

5. $\lambda^h(0_F) = \lambda^h(1_F) = 1_L$

The set all multi-fuzzy algebras of A over the multi-fuzzy field λ of F under under T-norm T with respect to the function h is denoted by $MFA^{(k,m)}(A, (\lambda, h \in L^M), T)$.
Here, k denotes the dimension of multi-fuzzy set in A ,m the dimension of multi-fuzzy field λ of F , and T the T- norm on L.
We say that $\mu = (\mu_{1,}, \mu_2, \dots, \mu_k)$ is a multi-fuzzy algebra of A over the multi-fuzzy field λ of F with respect to the function h,and denote it as:

$$\mu \epsilon MFA^{(k,m)}(A, (\lambda, h \in L^M), T).$$

The function h is called the bridge function of the space.

Theorem.4.2. Let A be an algebra over the field F, and let μ is a multi-fuzzy subset of A with $\mu(0_A) = 1_L$. Then:

If $\mu \in MFA^{(k,m)}(A, (\lambda, h \in L^M), T)$, then for all $x, y \epsilon A$ and $p, q \epsilon F$,

i) $\mu(px+qy) \geq \mathrm{T}\left(\mathrm{T}\left(\lambda^h(p)\mu(x)\right), \mathrm{T}\left(\lambda^h(q)), \mu(y)\right)\right)$

ii) $\mu(xy) \geq \mathrm{T}(\mu(x), \mu(y))$

Conversely, suppose the bridge function $h: M = \prod_{i=1}^m M_i \to L = \prod_{j=1}^k L_j$ satisfies $\lambda^h(0_F) = \lambda^h(1_F) = 1_L$ and that μ satisfies the conditions (i) and (ii), and assumes T is idempotent. Then

$$\mu \epsilon MFA^{(k,m)}(A,(\lambda, h \in L^M),T).$$

Proof : Let $\mu \epsilon MFA^{(k,m)}(A,(\lambda, h \in L^M),T)$

Then, by definition, for all $x, y \epsilon A$ and $for\ all\ q, p, \epsilon F$,we have:

i) $\mu\,(px+qy) \geq \mathrm{T}\,(\mu\,(px), \mu\,(qy)) \geq \mathrm{T}\Big(\mathrm{T}\left(\lambda^h(p)), \mu(x)\right), \mathrm{T}\left(\lambda^h(q), \mu(y)\right)\Big)$ and

ii) $\mu\,(xy) \geq \mathrm{T}(\,\mu(x), \mu(y))$

Conversely, suppose that multi-fuzzy subset μ of A satisfies the conditions (i) and (ii)

Set $p = 1_F = 1, q = 1_F = 1\ in\ (i)$.Then we get:

$\mu\,(x+y) \geq \mathrm{T}\Big(\mathrm{T}\left(\lambda^h(1_F), \mu(x)\right), \mathrm{T}\left(\lambda^h(1_F)), \mu(y)\right)\Big)$ ….(iii).

Consider:

$$\mathrm{T}\left(\lambda^h(1_F), \mu(x)\right) = \mathrm{T}\left(\mathrm{T}(1_L, \mu(x)), \mu(x)\right)$$
$$= \mathrm{T}(\,\mu(x), \mu(x))$$
$$= \mu(x)$$

Similarly, we can prove that $\mathrm{T}\left(\lambda^h(1_F), \mu(y)\right) = \mu(y)$

Substituting these results into (iii), we obtain:

$\mu\,(x+y) \geq \mathrm{T}(\,\mu(x), \mu(y))$.

Also, from (iii) ,we have:

$$\mu\,(px) = \mu\,(px + 0_F.x) \geq \mathrm{T}\Big(\mathrm{T}\left(\lambda^h(p), \mu(x)\right), \mathrm{T}\left(\lambda^h(0_F), \mu(x)\right)\Big)$$

Consider $\mathrm{T}\left(\lambda^h(0_F), \mu(x)\right) = \mathrm{T}(\,1_L, \mu(x))$

$$= \mu(x)$$

Therefore,

$$\mu\,(px) \geq \mathrm{T}\left(\mathrm{T}\left(\lambda^h(p), \mu(x)\right), \mu(x)\right)$$
$$= \mathrm{T}\left(\lambda^h(p), \mathrm{T}(\,\mu(x), \mu(x))\right)$$
$$= \mathrm{T}\left(\lambda^h(p), \mu(x)\right), \text{ since T is idempotent.}$$

$$\mu\,(-x) \geq \mathrm{T}\left(\lambda^h(-1_F), \mu(x)\right)$$
$$= \mathrm{T}\left(\lambda^h(1_F), \mu(x)\right) \text{ ,since } \lambda^h(-1_F) = h\big(\lambda(-1_F)\big) = h\big(\lambda(1_F)\big) = \lambda^h(1_F)$$
$$= \mathrm{T}(1_L, \mu(x))$$
$$= \mu(x) \quad \text{for all x}$$

Combining the condition (ii), we get:

$$\mu \epsilon MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$$

Theorem 4.3. Let A be an algebra over the field F, and let $\mu,\tau \epsilon MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$, where T is idempotent. Then the intersection $\mu \cap \tau \ \epsilon MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$

Proof : For any $x,y \ \epsilon \ A$ and $p \epsilon F$:

$(\mu \cap \tau)(x+y) = \mathrm{T}(\mu(x+y),\tau(x+y))$

$\geq \mathrm{T}\big(\mathrm{T}(\mu(x),\mu(y)),\mathrm{T}(\tau(x),\tau(y))\big)$

$= \mathrm{T}\big(\mathrm{T}(\mu(x),\ \tau(x)),\mathrm{T}(\mu(y),\tau(y))\big)$, by the commutativity and associativity of the operator T

$= \mathrm{T}((\mu \cap \tau)(x),(\mu \cap \tau)(y))$

Therefore, $(\mu \cap \tau)(x+y) \quad \geq \ \mathrm{T}((\mu \cap \tau)(x),(\mu \cap \tau)(y))$

Similarly, $(\mu \cap \tau)(px) \qquad = \mathrm{T}(\mu(px),\tau(px))$

$\geq \mathrm{T}\Big(\mathrm{T}\big(\lambda^h(p),\mu(x)\big),\mathrm{T}\big(\lambda^h(p),\tau(x)\big)\Big)$

$= \quad T\big(\mathrm{T}(\lambda^h(p),\lambda^h(p)),\mathrm{T}(\mu(x),\tau(x))\big)$, by the commutativity and associativity of the operator T, which simplifies to

$= \mathrm{T}\big(\lambda^h(p),(\mu \cap \tau)(x)\big)$

While $(\mu \cap \tau)(0_A) = \mathrm{T}(\mu(0_A),\tau(0_A)) = \mathrm{T}(1_L,1_L) = 1_L$,and

$(\mu \cap \tau)(xy) = \mathrm{T}(\mu(xy),\tau(xy))$

$\geq \mathrm{T}\big(\mathrm{T}(\mu(x),\mu(y)),\mathrm{T}(\tau(x),\tau(y))\big)$

$= \mathrm{T}\big(\mathrm{T}(\mu(x),\ \tau(x)),\mathrm{T}(\mu(y),\tau(y))\big)$ by the commutativity and associativity of the operator T

Hence, $\mu \cap \tau \ \epsilon MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$

Theorem 4.4. Let A be an algebra over the field F, and let $\{\mu_j \epsilon \ MFA^{(k,m)}(A,(\lambda,h \in L^M),T): j \in J\}$ be an indexed family ,where T is the minimum operator T-norm on L. Then their intersection:

$$\mu = \cap_{j \in J}\mu_j \ \epsilon MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$$

Proof: Let A be an algebra over the field F and let $x,y \ \epsilon \ A$ and $p \epsilon F$.Let $\wedge_{j \in J}$ denotes the infimum taken over all $j \in J$

$(\cap_{j \in J}\mu_j)(x+y) = \wedge_{j \in J}\{\mu_j(x+y): j \in J\}$

$\geq \ \wedge_{j \in J}T\{\mu_j(x),\ \mu_j(y)\}$

$= T\{\wedge_{j \in J}\mu_j(x),\ \wedge_{j \in J}\mu_j(y)\}$

$= \quad T\{(\cap_{j\in J}\mu_j)(x), (\cap_{j\in J}\mu_j)(y)\}$,

and similarly

$(\cap_{j\in J}\mu_j)(px) = \wedge_{j\in J}\{\mu_j(px)\}$

$\geq \wedge_{j\in J}\{\mu_j(px)\}$

$\geq \wedge_{j\in J}T\{\lambda^h(p), \mu_j(x))\}$

$= T\left\{\lambda^h(p), \left(\wedge_{j\in J}\mu_j(x)\right)\right\}$

$= T\{\lambda^h(p), (\cap_{j\in J}\mu_j)(x)\}$

Finally, $(\cap_{j\in J}\mu_j)(xy) = \wedge_{j\in J}\{\mu_j(xy)\}$

$\geq \wedge_{j\in J}T\{\mu_j(x), \mu_j(y)\}$

$= T\{\wedge_{j\in J}\mu_j(x), \wedge_{j\in J}\mu_j(y)\}$

$= T\{(\cap_{j\in J}\mu_j)(x), (\cap_{j\in J}\mu_j)(y)\}$

Therefore, $\mu = \cap_{j\in J}\mu_j \,\epsilon MFA^{(k,m)}(A, (\lambda, h \in L^M), T)$

Theorem 4.5. Let A be an algebra over the field F, and let B be a subset of A. Define the function $\mu: A \to L = \prod_{j=1}^{k} L_j$ by $\mu(x) = \begin{cases} 1_L, if\ x \in B \\ 0_L, if\ x \notin B \end{cases}$.

Let crisp function h be such that $\lambda^h(p) = 1_L\ for\ all\ p \in F$ and let T be idempotent. Then, $\mu\epsilon MFA^{(k,m)}(A, (\lambda, h \in L^M), T)$ if and only if B is a sub algebra of A.

Proof : Let B be a subset of A and $x, y \in B$, P$\in F$, and suppose $\mu\epsilon MFA^{(k,m)}(A, (\lambda, h \in L^M), T)$

Then $\mu(x) = 1_L, \mu(y) = 1_L$, so:

$\mu(px + y) \geq \mathrm{T}\left(\mathrm{T}\left(\lambda^h(p), \mu(x)\right), \mu(y)\right) = \mathrm{T}\left(\mathrm{T}(\lambda^h(p), 1_L), 1_L\right) = \mathrm{T}(\lambda^h(p), 1_L) = \lambda^h(p)$

Thus, $px + y \in B$ if $\lambda^h(p) = 1_L$ for all p.

Also, $\mu(xy) \geq \mathrm{T}(\mu(x), \mu(y)) = \mathrm{T}(1_L, 1_L) = 1_L$

That is, $xy \in B$

Hence ,B is a sub algebra of A if $\lambda^h(p) = 1_L\ for\ all\ p \in F$.

Conversely suppose that B is a sub algebra of A .Consider the multi-fuzzy set

$\mu(x) = \begin{cases} 1_L, if\ x \in B \\ 0_L, if\ x \notin B \end{cases}$.

For $x, y \in B$ and p$\in F$, we have $px + y \in B$

Then, $\mu(px + y) = 1_L = T(1_L, 1_L) = \mathrm{T}(\mathrm{T}(\lambda^h(p), 1_L), 1_L) = \mathrm{T}\left(\lambda^h(p), \mu(x)\right), \mu(y))$

and $\mu(xy) = 1_L = T(1_L, 1_L) = \mathrm{T}(\mu(x), \mu(y))$

Thus, $\mu \epsilon MFA^{(k,m)}(A,(\lambda, h \in L^M),T)$

Theorem.4.6. Let A and B be algebras over the field F, and let $f: A \rightarrow B$ be a linear transformation. Let $\mu_A \epsilon MFA^{(k,m)}(A,(\lambda, h \in L^M),T)$ be a multi-fuzzy algebra of A over λ.Then, $f(\mu_A)\epsilon\ MFA^{(k,m)}(B,(\lambda, h \in L^M),T)$. Assume $\prod_{j=1}^{k} L_j$ is the value domain of multi-fuzzy algebra of A over λ, with the T-norm in $\prod_{j=1}^{k} L_j$ as the infimum and the S-norm as the supremum. Assume that μ_A has the supremum property and h is monotonically increasing.

Proof : For $w_1, w_2 \in B$ and $p \in F$:

Let $f(\mu_A)(w_1)=$ $\sup\{\mu_A(u): u \in f^{-1}(w_1)\}= \mu_V(v_1)$,for some $v_1 \in A$

$f(\mu_A)(w_2)$ $= \sup\{\mu_A(u): u \in f^{-1}(w_2)\}= \mu_A(v_2)$, for some $v_2 \in A$

Then,

$$\begin{aligned}
f(\mu_A)(w_1+w_2) &= \sup\{\mu_A(u): u \in f^{-1}(w_1+w_2)\} \\
&\geq \sup\{\mu_A(u): u \in f^{-1}(w_1)+f^{-1}(w_2)\} \\
&\geq \mu_A(v_1+v_2), \text{ since } v_1+v_2 \in f^{-1}(w_1)+f^{-1}(w_2) \\
&\geq \inf\{\mu_A(v_1), \mu_A(v_2)\} \\
&= \inf\{(f(\mu_A)(w_1), f(\mu_A)(w_2))\}
\end{aligned}$$

Similarly,

$$\begin{aligned}
f(\mu_A)(pw_1) &= \sup\{\mu_A(u): u \in f^{-1}(pw_1) \\
&\geq \mu_A(pv_1), \text{ since } pv_1 \in f^{-1}(pw_1) \\
&\geq inf\left(\lambda^h(p), \mu_A(v_1)\right) \\
&= inf\left(\lambda^h(p), f(\mu_A)(w_1)\right).
\end{aligned}$$

For multiplication:

$$\begin{aligned}
f(\mu_A)(w_1w_2) &= \sup\{\mu_A(u): u \in f^{-1}(w_1w_2)\} \\
&\geq \sup\{\mu_A(u): u \in f^{-1}(w_1).f^{-1}(w_2)\} \\
&\geq \mu_A(v_1v_2), \text{ since } v_1v_2 \in f^{-1}(w_1).f^{-1}(w_2) \\
&\geq \inf\{\mu_A(v_1), \mu_A(v_2)\} \\
&= \inf\{(f(\mu_A)(w_1), f(\mu_A)(w_2))\}
\end{aligned}$$

Thus,

$$f(\mu_A)\epsilon\ MFA^{(k,m)}(B,(\lambda, h \in L^M),T)$$

Theorem 4.7. Let A and B be algebras over the field F, and let $f: A \rightarrow B$ be a linear transformation.Suppose $\mu_B \epsilon MFA^{(k,m)}(B,(\lambda, h \in L^M),T)$ is a multi-fuzzy algebra of B over λ.Then, $f^{-1}(\mu_B)\epsilon\ MFA^{(k,m)}(A,(\lambda, h \in L^M),T)$. Assume $\prod_{j=1}^{k} L_j$ is the value domain of multi-fuzzy algebra of A over λ, with the T-norm in $\prod_{j=1}^{k} L_j$ as the infimum and the S-norm as the supremum.

Proof : For $v_1, v_2 \in A$ and $p \in F$:

$$f^{-1}(\mu_B)(v_1+v_2) = \mu_B(f(v_1+v_2))$$
$$= \mu_B(f(v_1)+f(v_2))$$
$$\geq T(\ \mu_B(f(v_1)), \mu_B(f(v_2)))$$
$$= T(\ f^{-1}(\mu_B)(v_1), f^{-1}(\mu_B)(v_2)) .$$

Similarly,

$$f^{-1}(\mu_B)(pv_1) = \mu_B(f(pv_1))$$
$$= \mu_B(pf(v_1))$$
$$\geq T(\lambda^h(p)\ , \mu_B(f(v_1)))$$
$$= T(\lambda^h(p)\ ,\ f^{-1}(\mu_B)(v_1)) .$$

For multiplication:

$$f^{-1}(\mu_B)(v_1v_2) = \mu_B(f(v_1v_2))$$
$$= \mu_B(f(v_1).f(v_2))$$
$$\geq T(\ \mu_B(f(v_1)), \mu_B(f(v_2)))$$
$$= T(\ f^{-1}(\mu_B)(v_1), f^{-1}(\mu_B)(v_2))$$

Hence, $f^{-1}(\mu_B)\epsilon \in MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$

Theorem 4.8. Let A be an algebra over the field F .Let $\mu\,,\tau \,\epsilon\, MFA^{(k,m)}(A,(\lambda,h \in L^M),T)$, and suppose T is idempotent. Then the multi-fuzzy set $\mu \times \tau : A \times A \to L$ defined by

$(\mu \times \tau)(z_1,z_2) = \mathrm{T}\,\{\mu(z_1),\tau(z_2)\}$, for all $(z_1,z_2) \in A \times A$, belongs to $MFA^{(k,m)}(A \times A,(\lambda,h \in L^M),T)$.

Proof: Let $(z_1,z_2),(y_1,y_2) \in A \times A$ and $p \in F$. Then:

$$(\mu \times \tau)\big((z_1,z_2)+(y_1,y_2)\big) = (\mu \times \tau)\big((z_1+y_1,z_2+y_2)\big)$$
$$= \mathrm{T}\,\{\mu(z_1+y_1),\tau(z_2+y_2)\}$$
$$\geq \mathrm{T}\,\{\mathrm{T}\{\mu(z_1),\mu(y_1)\}\,,\mathrm{T}\{\tau(z_2),\tau(y_2)\}\}$$
$$= \mathrm{T}\,\{\mathrm{T}\{\mu(z_1),\tau(z_2)\}\,,T\{\mu(y_1),\tau(y_2)\}$$
$$= \mathrm{T}\,\{(\mu \times \tau)\big((z_1,z_2)\big),(\mu \times \tau)\big((y_1,y_2)\big)$$

For scalar multiplication:

$$(\mu \times \tau)\big(p(z_1,z_2)\big) = (\mu \times \tau)\big((pz_1,pz_2)\big)$$
$$= \mathrm{T}\,\{\mu(pz_1),\tau(pz_2)\}$$
$$\geq \mathrm{T}\,\Big\{\mathrm{T}\{\lambda^h(p),\mu(z_1)\}\,,\mathrm{T}\{\{\lambda^h(p),\tau(z_2)\}\Big\}$$
$$= \mathrm{T}\,\Big\{\mathrm{T}\{\lambda^h(p),\lambda^h(p)\}\,,\mathrm{T}\{\{\mu(z_1),\tau(z_2)\}\Big\}$$

$= T\{\lambda^h(p), (\mu \times \tau)(z_1, z_2)\}$ for all $p \epsilon F$, $(z_1, z_2) \in A \times A$

$(\mu \times \tau)(-(z_1, z_2)) \geq T\{\lambda^h(-1), (\mu \times \tau)(z_1, z_2)$

$= T\{1, (\mu \times \tau)(z_1, z_2)$

$= (\mu \times \tau)(z_1, z_2)$, for all $(z_1, z_2) \in A \times A$

$(\mu \times \tau)(0,0) = T\{\mu(0), \tau(0)\} = Tf\{1_L, 1_L\} = 1_L$

$$(\mu \times \tau)((z_1, z_2).(y_1, y_2)) = (\mu \times \tau)((z_1 y_1, z_2 y_2))$$
$$= \mathrm{T}\{\mu(z_1 y_1), \tau(z_2 y_2)\}$$
$$\geq \mathrm{T}\{\mathrm{T}\{\mu(z_1), \mu(y_1)\}, \mathrm{T}\{\tau(z_2), \tau(y_2)\}\}$$
$$= \mathrm{T}\{\mathrm{T}\{\mu(z_1), \tau(z_2)\}, T\{\mu(y_1), \tau(y_2)\}$$
$$= \mathrm{T}\{(\mu \times \tau)((z_1, z_2)), (\mu \times \tau)((y_1, y_2))$$

Thus, for all $(z_1, z_2), (y_1, y_2) \in A \times A$ and $p \in F$,we have
$$(\mu \times \tau)((z_1, z_2).(y_1, y_2)) = (\mu \times \tau)((z_1 y_1, z_2 y_2))$$
$$= \mathrm{T}\{\mu(z_1 y_1), \tau(z_2 y_2)\}$$
$$\geq \mathrm{T}\{\mathrm{T}\{\mu(z_1), \mu(y_1)\}, \mathrm{T}\{\tau(z_2), \tau(y_2)\}\}$$
$$= \mathrm{T}\{\mathrm{T}\{\mu(z_1), \tau(z_2)\}, T\{\mu(y_1), \tau(y_2)\}$$
$$= \mathrm{T}\{(\mu \times \tau)((z_1, z_2)), (\mu \times \tau)((y_1, y_2))$$

Hence , $\mu \times \tau \in MFA^{(k,m)}(A \times A, (\lambda, h \in L^M), T)$

4. Conclusions: This paper has explored the structure and properties of multi-fuzzy algebras over multi-fuzzy fields, extending the traditional framework of fuzzy algebra to multi-dimensional membership functions. By introducing a bridge function h, we have provided a flexible approach to defining multi-fuzzy algebras when the dimensions of the multi-fuzzy set on the algebra and the multi-fuzzy field differ. The key findings include the characterization of multi-fuzzy algebras, their intersections, and their behaviour under linear transformations. These results lay a solid foundation for further research in multi-fuzzy linear algebra.

Future work could explore the role of different bridge functions in determining the properties of multi-fuzzy algebras, as well as their applications in decision-making, pattern recognition, and artificial intelligence. Additionally, the relationship between multi-fuzzy algebras and other algebraic structures such as multi-fuzzy rings and multi-fuzzy groups could be investigated to develop a more comprehensive theory

References:

[1]Ramakrishnan, T. V., & Sebastian, S. (2010). A study on multi-fuzzy sets. *International Journal of Applied Mathematics, 23*(4), 713-721. Academic Publications.

[2]. Sebastian, S., & Ramakrishnan, T. V. (2010). Multi-fuzzy sets. *International Mathematical Forum, 5*(50), 2471-2476. Hikari Ltd.

[3]. Sebastian, S., & Ramakrishnan, T. V. (2010). Multi-fuzzy sets: An extension of fuzzy sets *Fuzzy Information and Engineering, 4*, 389-397. Springer.

[4] Katsaras A. K., Liu D.B., Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces, Journal of Mathematical Analysis and Applications, 58, 135-146, (1977)

[5] Abdukhalikov, K. S., The Dual of Fuzzy Subspace, Fuzzy Sets and Systems, 82, 375-381, (1996)

[6].S. Nanda, Fuzzy linear spaces over valued fields, Fuzzy Sets and Systems, 42(3)(1991), 351-354.

[7] Rasul rasuli , Fuzzy vector spaces under norms ,*Annals of Mathematics and Computer Science* . ISSN: 2789-7206, Vol 9 (2022) 7-24