A Structural Analysis of FMRA in $L^2(\mathrm{R})^n$

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Abstract

Frame Multiresolution Analysis (FMRA) extends the classical concept of multiresolution analysis to the context of frames in Hilbert spaces. In this paper, we investigate FMRA in the superspace $L^2(R)^n$ and establish that the frame property of the system $\{T^k\phi_1 \oplus \cdots \oplus \phi_n : k \in Z\}$ in V_0 is preserved under dilations by the operator U_C. Specifically, we prove that the dilated-translated system $\{U^j\ T^k\phi_1 \oplus \cdots \oplus \phi_n : k \in Z\}$ forms a frame for V_j with the same frame bounds as in V_0. This result demonstrates the stability of FMRA frames under generalized dilation and translation operators in $L^2(R)^n$, facilitating their application to multiple signal processing and superwavelet constructions.

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. 1. Introduction

Wavelet theory owes much of its development to the pioneering work of Daubechies. A central aspect that propelled its success is multiresolution analysis (MRA), which provides a robust framework for constructing orthonormal wavelet bases with strong localization features, proving invaluable across applications in physics and engineering. In a similar spirit, the concepts of frames and Frame MRA (FMRA) in $L^2(\mathrm{R})$ have evolved substantially, driven by their relevance in fields such as speech and image processing.

2. Basic Definitions and Theoretical Background

We begin with the definition of frames, wavelet and frame wavelet.

Definition 2.1. *A countable collection $\{u_n\}$ of vectors in a Hilbert space H is a* frame *if it satisfies the following property: There exist constants $A, B > 0$ such that for $x \in H$*

$$A||x||^2 \leq \sum_n |< x, u_n >|^2 \leq B||x||^2.$$

The largest value of A satisfying the frame inequality is referred to as the lower frame bound, while the smallest admissible value of B is known as the upper frame bound. When these two bounds are equal, the frame is called a tight frame. In particular, if $A = B = 1$, the frame is termed a normalized tight frame.

Definition 2.2. *An element $f \in L^2(\mathrm{R})$ is called an* orthonormal wavelet *(respectively, a* frame wavelet*) if the collection $\{U^j T^k f : j, k \in \mathrm{Z}\}$ forms an orthonormal basis (respectively, a frame) for $L^2(\mathrm{R})$. Here, T and U denote the translation and dilation unitary operators on $L^2(\mathrm{R})$, defined by $(Tf)(t) = f(t - 1)$ and $(Uf)(t) = 2^{1/2}f(2t)$.*

Definition 2.3. *A* frame multiresolution analysis (FMRA) *of $L^2(\mathrm{R})$ is a pair $\{V_j, f\}_{j\in \mathrm{Z}}$, where $\{V_j\}$ is an increasing sequence of closed subspaces of $L^2(\mathrm{R})$ and $f \in V_0$, satisfying the following conditions:*

(1) The union $\cup_j V_j$ is dense in $L^2(\mathrm{R})$, that is, $\overline{\cup_j V_j} = L^2(\mathrm{R})$.

(2) For each $j \in \mathrm{Z}$, a function g belongs to V_j if and only if its dilate Ug belongs to V_{j+1}, that is, $g \in V_j \Leftrightarrow Ug \in V_{j+1}$.

(3) The space V_0 is invariant under integer translations; that is, for all $k \in \mathrm{Z}$, $g \in V_0 \Rightarrow T^k g \in V_0$.

(4) The set $\{T^k f : k \in \mathrm{Z}\}$ forms a frame for V_0.

Here, T and U denote the translation and dilation operators, respectively.

The function *f* defined above is referred to as a *scaling function* for the FMRA.

Frame wavelets associated with FMRA do not exist in the standard sense in $L^2(\mathbb{R})^n$ [2]. In other words, when dealing with multiple signals, simply extending the construction techniques for frame wavelets in $L^2(\mathbb{R})$ does not suffice. However, it has been shown that frame wavelets can be constructed for multiple signals if certain modifications are made to the usual dilation and translation operators [6]. To achieve this, we define the translation and dilation operators on $L^2(\mathbb{R})^n$ as follows:

$$T_C(f_1 \oplus \cdots \oplus f_n) = z_1 T f_1 \oplus \cdots \oplus z_n T f_n,$$

and

$$U_C(f_1 \oplus \cdots \oplus f_n) = U f_2 \oplus \cdots \oplus U f_n \oplus U f_1,$$

where T and U denote the translation and dilation operators on $L^2(\mathbb{R})$ as defined earlier. The scalars $z_1, \ldots, z_n$ are chosen to satisfy

$$z_1^2 = z_2\ ,\ \ z_2^2 = z_3\ ,\ \ z_{n-1}^2 = z_n\ , \ldots,\ \ z_n^2 = z_1\ .$$

Definition 2.4. *A* frame multiresolution analysis (FMRA) *of* $L^2(\mathbb{R})^n$ *is a collection* $\{V_j, \phi_1 \oplus \cdots \oplus \phi_n\}_{j\in \mathbb{Z}}$, *where* $\{V_j\}$ *is an increasing sequence of closed subspaces of* $L^2(\mathbb{R})^n$ *and* $\phi_1 \oplus \cdots \oplus \phi_n \in V_0$, *satisfying the following conditions:*

(1) The union $\bigcup_j V_j$ *is dense in* $L^2(\mathbb{R})^n$, *that is,* $\overline{\bigcup_j V_j} = L^2(\mathbb{R})^n$.
(2) For each $j \in \mathbb{Z}$,

$$f_1 \oplus \cdots \oplus f_n \in V_j \Leftrightarrow U_C(f_1 \oplus \cdots \oplus f_n) \in V_{j+1}.$$

(3) The space V_0 *is invariant under integer powers of the operator* T_C; *that is, for all* $k \in \mathbb{Z}$,

$$f_1 \oplus \cdots \oplus f_n \in V_0 \Rightarrow T_C^k(f_1 \oplus \cdots \oplus f_n) \in V_0.$$

(4) The set

$$(\phi_1 \oplus \cdots \oplus T^k(\phi_n) : k \in \mathbb{Z}\}$$

forms a frame for V_0.

3. Stability of FMRA Frames under Dilation and Translation Operators

Proposition 3.1. *Let* $\{V_j, \phi_1 \oplus \cdots \oplus \phi_n\}_{j\in \mathbb{Z}}$ *be an FMRA of* $L^2(\mathbb{R})^n$. *If* $\{T_C^k\phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\}$ *is a frame for* V_0 , *then the system*

$\{U_C^j T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\}$ *is a frame for* V_j *with the same frame bounds.*

Proof. We establish the result for the case $n = 2$, noting that the argument extends naturally to the general case.

$\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ is a frame for V_0 implies that there exist lower frame bound, say, $A > 0$ and upper frame bound , say, $B < \infty$ such that for all $f_1 \oplus f_2 \in V_0$

$$A\left||f_1 \oplus f_2\ |\right|^2 \le \sum_k \left|< f_1 \oplus f_2, T_C^k\phi_1 \oplus \phi_2\right|^2 \le B\left||f_1 \oplus f_2\ |\right|^2.$$

Also,

$$\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subseteq V_0 \Leftrightarrow \{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subseteq V_j.$$

$$\begin{aligned}
\langle U_C f_1 \oplus f_2, U_C f1 \oplus f2\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} &= \langle U f_2 \oplus U f_1,\ U f_2 \oplus U f_1\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&= \langle U f_2,\ U f_2\rangle_{L^2(\mathbb{R})} + \langle U f_1,\ U f_1\rangle_{L^2(\mathbb{R})} \\
&= \langle f_2,\ f_2\rangle_{L^2(\mathbb{R})} + \langle f_1,\ f_1\rangle_{L^2(\mathbb{R})}\ ,
\end{aligned}$$

as U is unitary operator on $L^2(\mathbb{R})$

$$= \langle f_1, f_1\rangle_{L^2(\mathrm{R})} + \langle f_2, f_2\rangle_{L^2(\mathrm{R})}$$

$$= \langle f_1 \oplus f_2, f_1 \oplus f_2\rangle_{L^2(\mathrm{R})\oplus L^2(\mathrm{R})} .$$

Hence U_C is a unitary operator on $L^2(\mathrm{R}) \oplus L^2(\mathrm{R})$, and hence

$$U_C U_{C^*} = I = U_{C^*} U_C$$

which implies that

$$U_{C^*} = U_C^{-1}.$$

In general, $(U_C^j)^* = U_C^{-j}$ for $j \in \mathrm{Z}$.

Hence for $f_1 \oplus f_2 \in V_j$,

$$\sum_k \left|< f_1 \oplus f_2, U_C^j T_C^k \phi_1 \oplus \phi_2 >\right|^2 = \sum_k \left|< U_C^{-j} f_1 \oplus f_2, T_C^k \phi_1 \oplus \phi_2 >\right|^2$$

Since $U_C^{-j} f_1 \oplus f_2 \in V_0$ and $\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathrm{Z}\}$ is a frame for V_0, we have

$$A|| U_C^{-j} f_1 \oplus f_2||^2 \le \sum_k \left|< U_C^{-j} f_1 \oplus f_2, T_C^k \phi_1 \oplus \phi_2 >\right|^2 \le B \left|\left| U_C^{-j} f_1 \oplus f_2\right|\right|^2.$$

But,

$$|| U_C^{-j} f_1 \oplus f_2||^2 = < U_C^{-j} f_1 \oplus f_2, U_C^{-j} f_1 \oplus f_2 >_{L^2(\mathrm{R})\oplus L^2(\mathrm{R})}$$

$$= < (U_C^{-j})^* U_C^{-j} f_1 \oplus f_2, f_1 \oplus f_2 >_{L^2(\mathrm{R})\oplus L^2(\mathrm{R})}$$

$$= < f_1 \oplus f_2, f_1 \oplus f_2 >_{L^2(\mathrm{R})\oplus L^2(\mathrm{R})}$$

$$= \left||f_1 \oplus f_2 |\right|^2.$$

Now, we have

$$A\left||f_1 \oplus f_2 |\right|^2 \le \sum_k \left|< f_1 \oplus f_2, T_C^k \phi_1 \oplus \phi_2\right|^2 \le B\left||f_1 \oplus f_2 |\right|^2.$$

Hence $\{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathrm{Z}\}$ *is a frame for* V_j *with the same frame bounds as the frame* $\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathrm{Z}\}$ *of* V_0.

4. Conclusion

In this paper, we explored the framework of Frame Multiresolution Analysis (FMRA) in the superspace $L^2(\mathrm{R})^n$. We established that if $\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathrm{Z}\}$ forms a frame for V_0, then the dilated and translated system $\{U_C^j T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathrm{Z}\}$ also forms a frame for V_j with the same frame bounds, demonstrating stability under the generalized dilation operator U_C. The result enhances the theoretical foundation of FMRA and paves the way for applications in multiple signal processing and superwavelet analysis.

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